Chapter 1: Functions

Contents

1 Functions, Domain/Codomain/Range

Definition 1.1 (Function)**.** Let A and B be sets. A *function* from A to B is a rule (more precisely, a relation) f that assigns to each element $x \in A$ *exactly one* element $y \in B$. We write

$$f : A \to B, \qquad y = f(x).$$

The statement "$y = f(x)$" means that y is the unique output in B associated with the input x in A.

Definition 1.2 (Domain, codomain, range)**.** For a function $f : A \to B$:

- The *domain* is the input set: $\mathrm{Dom}(f) = A$.
- The *codomain* is the target set: $\mathrm{Cod}(f) = B$.
- The *range* (or *image*) is the set of actual outputs:

$$\mathrm{Ran}(f) = \{\, f(x) \in B : x \in A \,\} \subseteq B.$$

Remark (Independent vs. dependent variables)*.* In calculus, one typically models a dependence of a quantity y on another quantity x. Formally:

- x is an *independent variable*: it is chosen freely from the domain.
- y is a *dependent variable*: its value is determined by the function rule $y = f(x)$.

The function f itself is not a number; it is the mapping (the assignment mechanism) that takes x and returns y.

A quick catalogue

The table below lists common examples with explicit domains and ranges.

Function $f(x)$	**Domain**	**Range**
x^2	$\mathbb{R}$	$[0, \infty)$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\dfrac{1}{x-2}$	$\mathbb{R} \setminus \{2\}$	$\mathbb{R} \setminus \{0\}$
$\sin x$	$\mathbb{R}$	$[-1, 1]$
$\lvert x \rvert$	$\mathbb{R}$	$[0, \infty)$

2 Graphs and the Vertical Line Test

Definition 2.1 (Graph of a function)**.** If $f : A \to B$ with $A, B \subseteq \mathbb{R}$, the *graph* of f is the set

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : x \in A,\ y = f(x)\}.$$

Equivalently, it is the set of all points $(x, f(x))$ in the xy-plane.

2.1 Why "two outputs for one input" fails

A key requirement in the definition of a function is *uniqueness of the output.*

Example 2.1 (Circle relation vs. function)**.** Consider the circle relation

$$x^2 + y^2 = 1.$$

For a given $x \in (-1, 1)$, there are generally two possible y values:

$$y = \sqrt{1 - x^2} \quad \text{and} \quad y = -\sqrt{1 - x^2}.$$

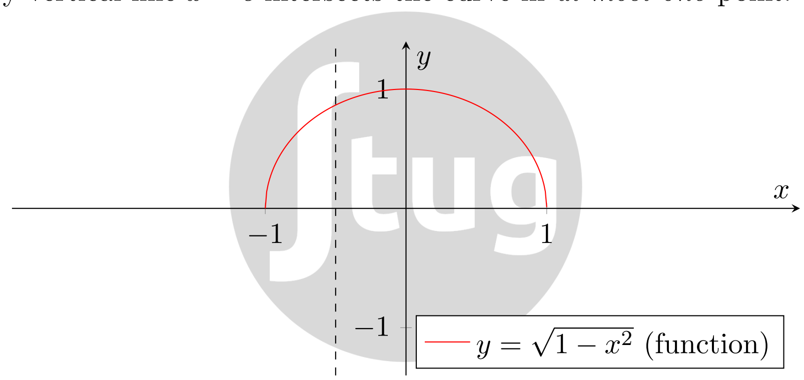
Thus, the circle equation defines a *relation* in the plane, but *not* a function $y = f(x)$ on $(-1, 1)$, because one input x corresponds to two outputs.

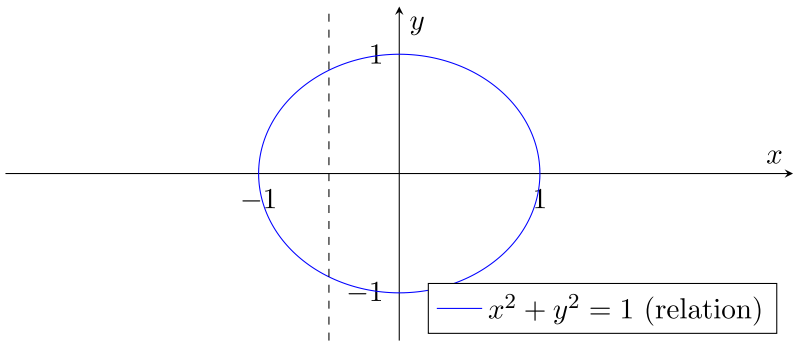
However, if we *restrict* to one branch, then we do obtain a function:

$$f(x) = \sqrt{1 - x^2}, \quad \mathrm{Dom}(f) = [-1, 1], \quad \mathrm{Ran}(f) = [0, 1].$$

2.2 Vertical line test

Definition 2.2 (Vertical line test)**.** A curve in the plane is the graph of a function $y = f(x)$ if and only if every vertical line $x = c$ intersects the curve in *at most one* point.



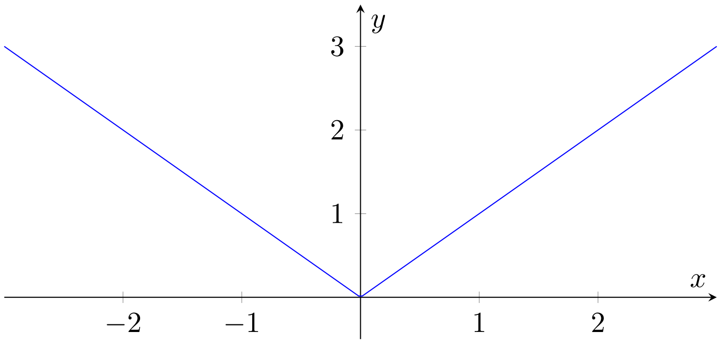


3 Piecewise-Defined Functions; Floor and Ceiling

Definition 3.1 (Piecewise-defined function)**.** A *piecewise-defined function* is specified by different formulas on different parts of the domain. Formally, we partition the domain into subsets and define f on each subset, ensuring the definitions are consistent on overlaps.

Example 3.1 (Absolute value)**.**

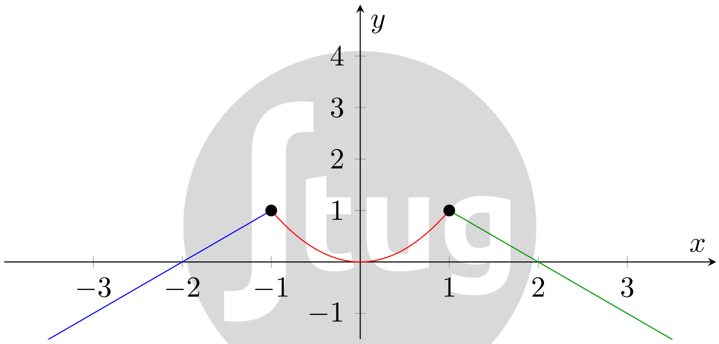
$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$



Example 3.2 (A custom piecewise function)**.** Define

$$f(x) = \begin{cases} x+2, & x < -1, \\ x^2, & -1 \le x \le 1, \\ 2-x, & x > 1. \end{cases}$$

This function is well-defined because each x belongs to exactly one case (including the endpoints).



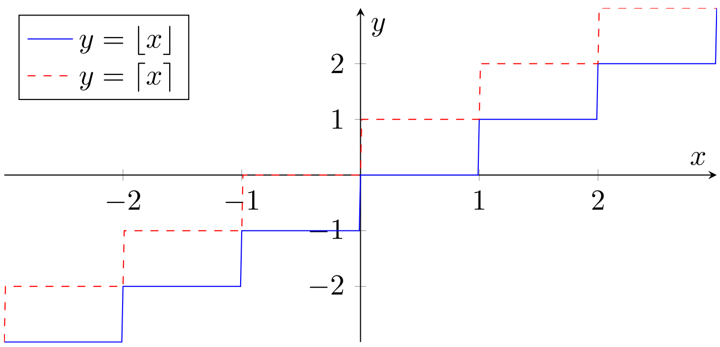
3.1 Floor and ceiling functions

Definition 3.2 (Floor and ceiling)**.** For $x \in \mathbb{R}$:

$$\lfloor x \rfloor = \text{the greatest integer } \le x, \qquad \lceil x \rceil = \text{the least integer } \ge x.$$

Example 3.3.

$$\lfloor 2.7 \rfloor = 2, \quad \lceil 2.7 \rceil = 3, \qquad \lfloor -1.2 \rfloor = -2, \quad \lceil -1.2 \rceil = -1.$$



4 Increasing/Decreasing; Even/Odd; Symmetry

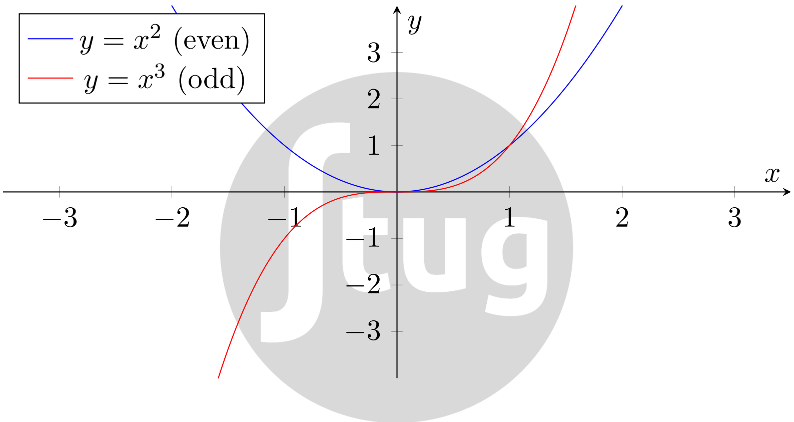
Definition 4.1 (Increasing and decreasing)**.** Let I be an interval and $f : I \to \mathbb{R}$.

- f is *increasing* on I if for all $x_1 < x_2$ in I, we have $f(x_1) \leq f(x_2)$.
- f is *strictly increasing* if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- f is *decreasing* on I if $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$.
- f is *strictly decreasing* if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Definition 4.2 (Even and odd functions)**.** Let f be defined on a domain symmetric about 0 (i.e., if x is in the domain, so is $-x$).

- f is *even* if $f(-x) = f(x)$ for all x in its domain.
- f is *odd* if $f(-x) = -f(x)$ for all x in its domain.

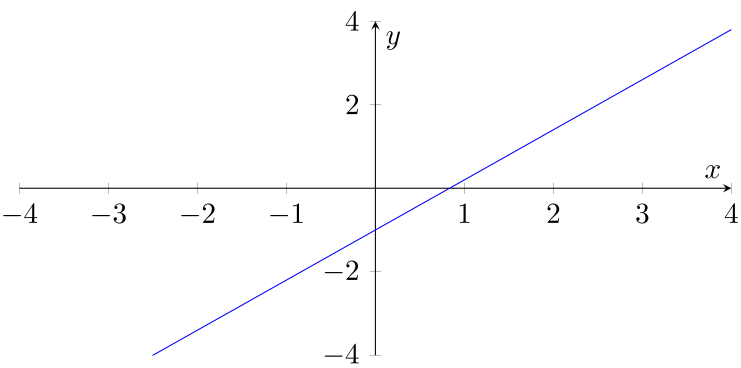
Remark (Symmetry)*.* If f is even, its graph is symmetric about the y-axis. If f is odd, its graph is symmetric about the origin (a 180° rotation about $(0, 0)$).



5 Common Function Types (with typical graphs)

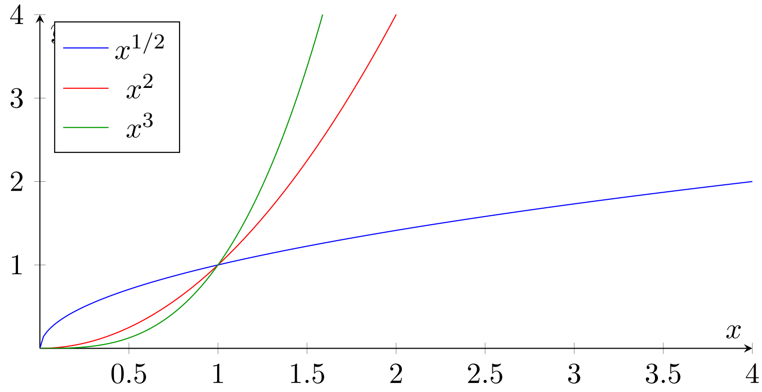
5.1 Linear functions

A *linear* (affine) function has the form $f(x) = mx + b$. The slope m measures rate of change, and b is the y-intercept.



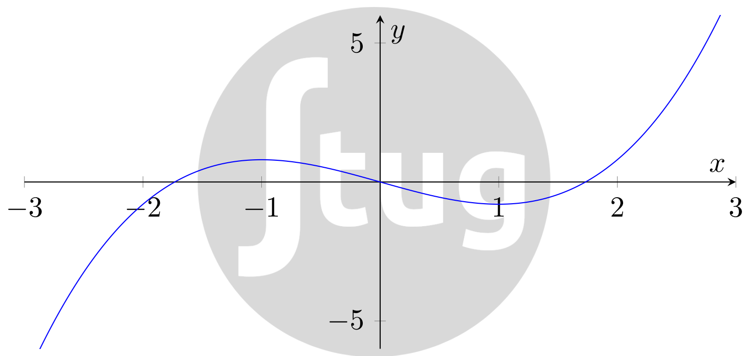
5.2 Power functions

A *power function* has the form $f(x) = x^a$ (for real exponent a) on a suitable domain (often $x > 0$ if a is not an integer).



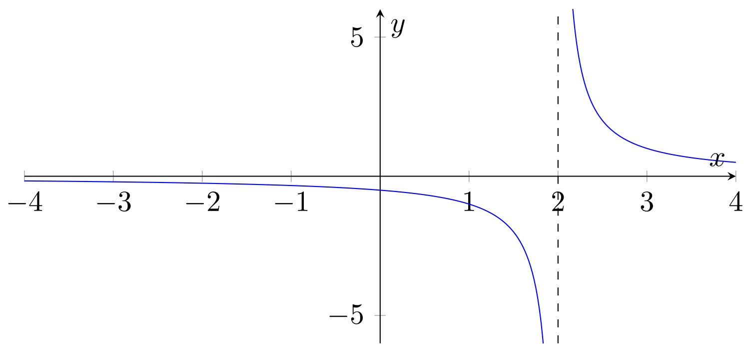
5.3 Polynomials

A *polynomial* is $p(x) = a_n x^n + \cdots + a_1 x + a_0$ with real coefficients. Polynomials are defined for all real x and are continuous and smooth.



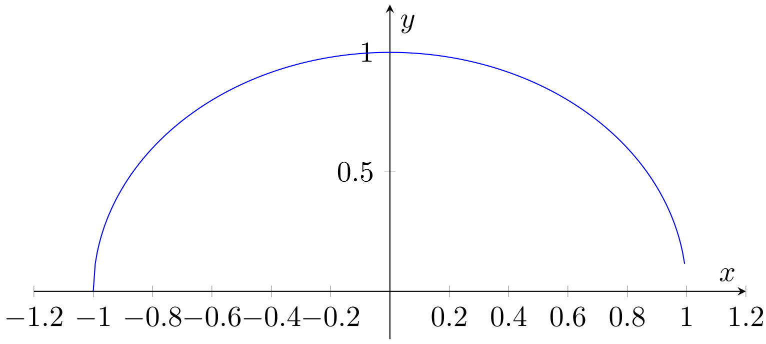
5.4 Rational functions

A *rational function* is a quotient $r(x) = \dfrac{p(x)}{q(x)}$ of polynomials, with domain excluding zeros of $q(x)$ (vertical asymptotes may occur).



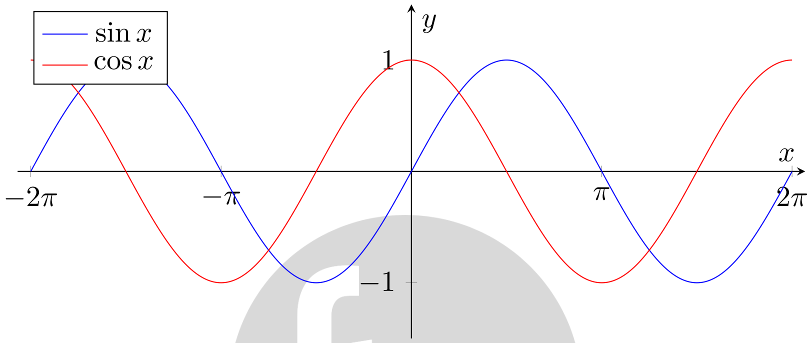
5.5 Algebraic functions

An *algebraic function* is built from polynomials using algebraic operations and roots (e.g. $\sqrt{1 - x^2}$). Domains are constrained to keep expressions real.



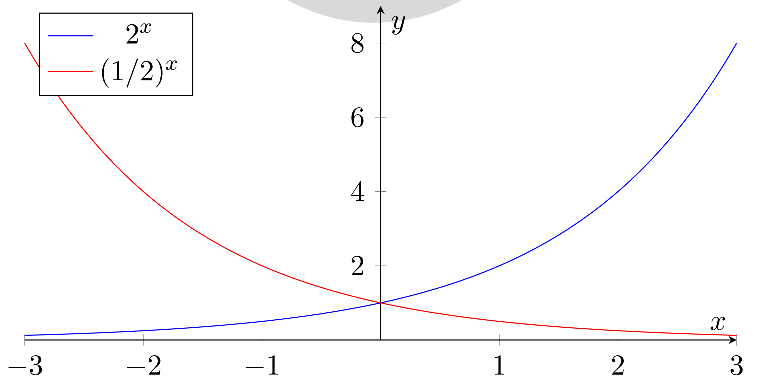
5.6 Trigonometric functions

Trigonometric functions (like $\sin x$ and $\cos x$) are periodic and arise naturally in oscillations, waves, and rotations.



5.7 Exponential functions

An exponential function has the form $f(x) = a^x$ with $a > 0$ and $a \neq 1$. It models growth/decay processes.

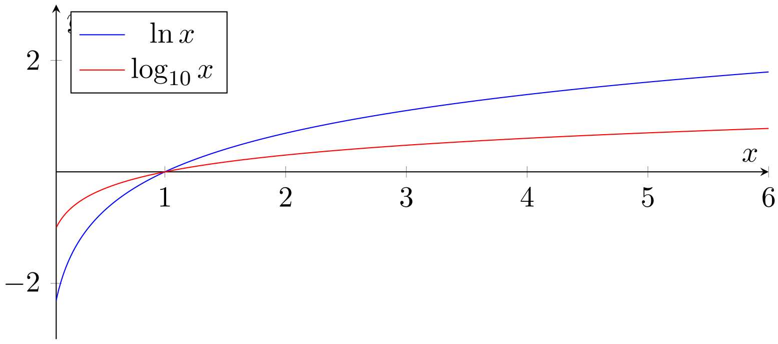


5.8 Logarithmic functions

The logarithm $\log_a x$ is the inverse of a^x:

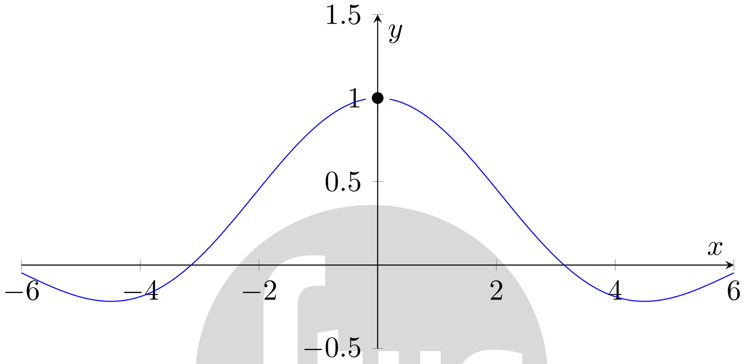
$$\log_a x = y \iff a^y = x, \qquad a > 0,\ a \neq 1.$$

Its domain is $(0, \infty)$.



5.9 Transcendental functions

A function is often called *transcendental* if it is not algebraic; typical examples include exponentials, logarithms, and many combinations such as e^{x^2} or $\sin x/x$.



6 Composite Functions

Definition 6.1 (Composition)**.** Let $g : A \to B$ and $f : B \to C$. The *composition* $f \circ g$ is the function $A \to C$ defined by

$$(f \circ g)(x) = f(g(x)).$$

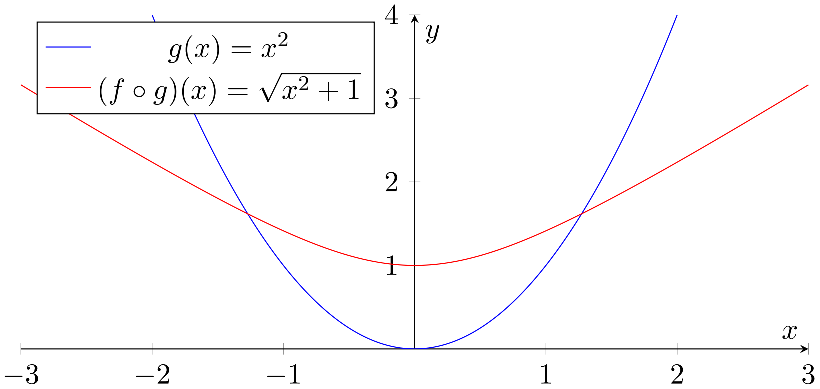
The composition is defined only when the output of g lies in the domain of f.

Example 6.1. Let $g(x) = x^2$ and $f(u) = \sqrt{u+1}$ (domain $u \geq -1$). Then

$$(f \circ g)(x) = \sqrt{x^2+1}, \qquad (g \circ f)(x) = \left(\sqrt{x+1}\right)^2 = x+1,$$

with the domain of $g \circ f$ being $x \geq -1$.

x	$g(x) = x^2$	$(f \circ g)(x) = \sqrt{x^2+1}$
-2	4	$\sqrt{5}$
-1	1	$\sqrt{2}$
0	0	1
1	1	$\sqrt{2}$
2	4	$\sqrt{5}$



7 Shifting, Scaling, and Reflection

Graph transformations are best understood as changes to the input variable (x) and the output value (y).

7.1 Shifts

- **Vertical shift:** $y = f(x) + k$ shifts the graph of f up by k (down if $k < 0$).
- **Horizontal shift:** $y = f(x - h)$ shifts the graph of f right by h (left if $h < 0$).

7.2 Scaling

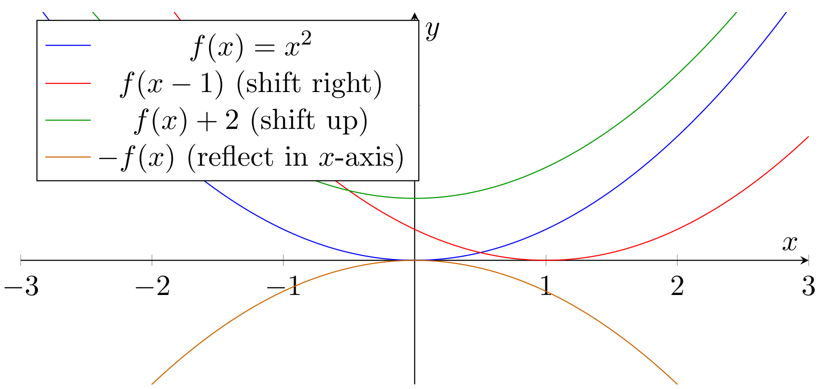
- **Vertical scaling:** $y = af(x)$ stretches by factor $|a|$ vertically (and flips across the x-axis if $a < 0$).
- **Horizontal scaling:** $y = f(bx)$ compresses horizontally by factor $|b|$ (stretches if $|b| < 1$).

7.3 Reflections

- **Across the x-axis:** $y = -f(x)$.
- **Across the y-axis:** $y = f(-x)$.

Example 7.1 (Transformations of $f(x) = x^2$)**.** Consider $f(x) = x^2$ and the transformed functions:

$$y = f(x-1) = (x-1)^2, \qquad y = f(x)+2 = x^2+2, \qquad y = -f(x) = -x^2, \qquad y = f(2x) = (2x)^2 = 4x^2.$$



8 Trigonometric Functions (Angle, Unit Circle, Periodicity, Identities)

8.1 Angles and radian measure

An *angle* θ can be measured in degrees or radians. The radian measure is defined geometrically:

$$\theta = \frac{\text{arc length}}{\text{radius}}$$

on a circle. Thus, one full revolution corresponds to arc length $2\pi r$ on radius r, giving 2π radians.

8.2 Six basic trigonometric functions

On the unit circle, a point corresponding to angle θ is $(\cos\theta, \sin\theta)$. The six trigonometric functions are:

$$\sin\theta,\ \cos\theta,\ \tan\theta = \frac{\sin\theta}{\cos\theta},\ \csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta},$$

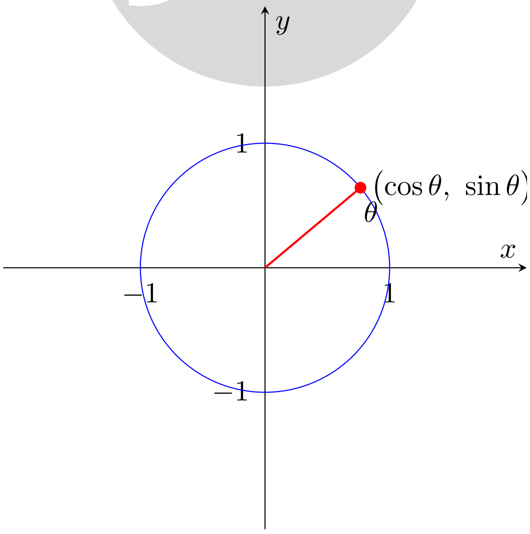
with domains restricted where denominators vanish.

8.3 Unit circle and right triangles

For acute angles in right triangles, if θ is an acute angle, then

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

The unit circle extends these definitions to all real angles by interpreting $(\cos\theta, \sin\theta)$ as coordinates on the circle.



8.4 Periodicity

Definition 8.1 (Period)**.** A function f is *periodic* if there exists $T > 0$ such that for all x in the domain of f,

$$f(x + T) = f(x).$$

The smallest such positive T (if it exists) is called the *fundamental period.*

For example, $\sin x$ and $\cos x$ have fundamental period 2π, while $\tan x$ has period π.

8.5 Core trigonometric identities

Some identities used constantly in calculus and engineering:

- Pythagorean: $\sin^2\theta + \cos^2\theta = 1$.
- Angle addition:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

- Double-angle (special case):

$$\sin(2\theta) = 2\sin\theta\cos\theta, \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1.$$

8.6 Transformations of trigonometric graphs

A general sinusoid is

$$y = A\sin\big(\omega(x-h)\big) + k$$

(or with cos). Interpretations:

- $|A|$ = amplitude (vertical stretch); sign of A reflects in the x-axis.
- Period $T = \dfrac{2\pi}{|\omega|}$.
- h = horizontal shift; k = vertical shift (midline $y = k$).