Chapter 2: Limits, Continuity, and the Road to Derivatives

Contents

1 Rates of Change and Tangent Lines

1.1 Average speed and average rate of change

Let $s(t)$ be the position of an object (meters) at time t (seconds). The **average speed** on $[t_1, t_2]$ is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

For a general function $y = f(x)$, the **average rate of change** on $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which is the slope of the **secant line** through $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Example 1.1 (Average speed from data)**.** Suppose an object's position is $s(t) = 5t+2$ (meters). Compute the average speed on $[3, 8]$:

$$\frac{s(8) - s(3)}{8 - 3} = \frac{(5 \cdot 8 + 2) - (5 \cdot 3 + 2)}{5} = \frac{40 - 15}{5} = 5.$$

Example 1.2 (Average speed that changes)**.** Let $s(t) = t^2$ (meters). Average speed on $[1, 1+h]$:

$$\frac{s(1 + h) - s(1)}{h} = \frac{(1 + h)^2 - 1}{h} = 2 + h.$$

As $h \to 0$, this approaches 2.

h	$\frac{s(1+h) - s(1)}{h}$	value
1	$2 + 1$	3
0.5	$2 + 0.5$	2.5
0.1	$2 + 0.1$	2.1
0.01	$2 + 0.01$	2.01
0.001	$2 + 0.001$	2.001

1.2 Secant slopes and the tangent line idea

For a curve $y = f(x)$, fix $P = (a, f(a))$ and let $Q = (a + h, f(a + h))$ move on the curve. The secant slope is

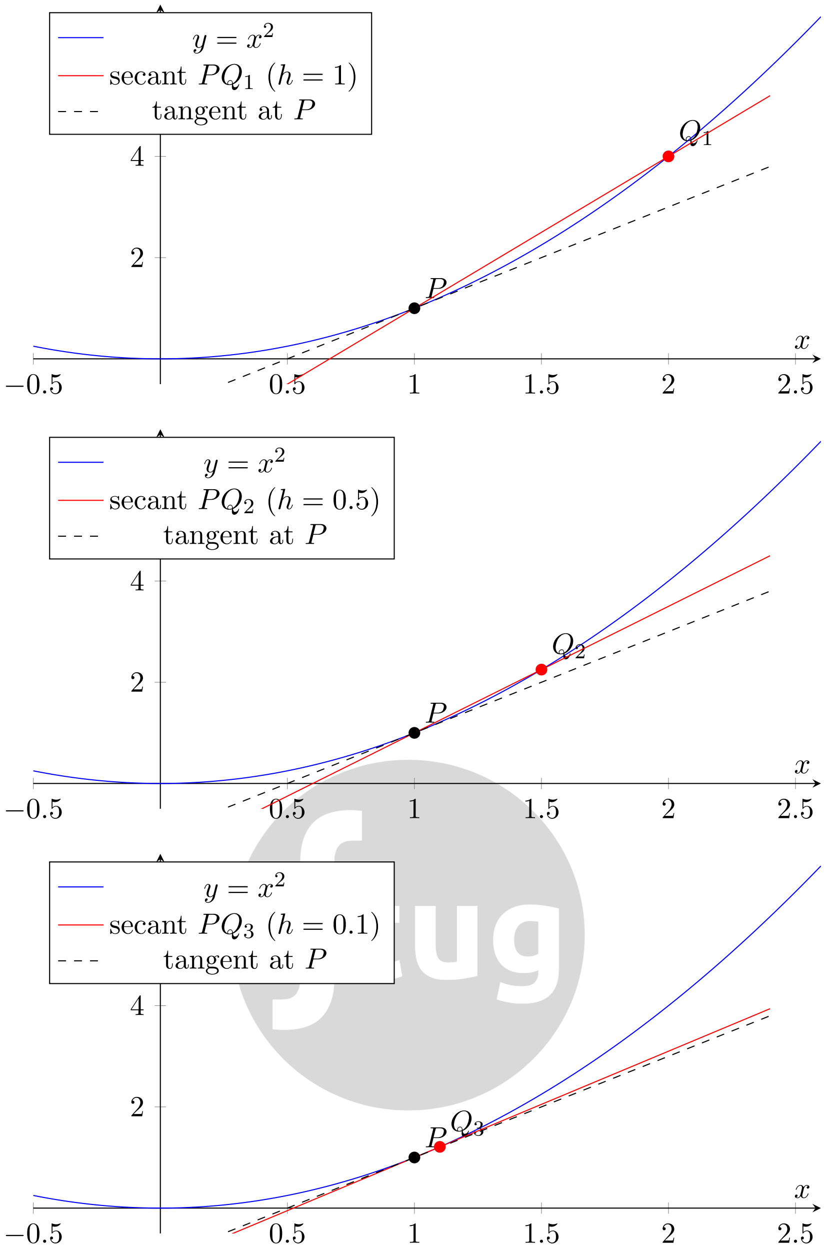
$$m_{PQ} = \frac{f(a + h) - f(a)}{h}.$$

As $Q \to P$ (i.e. $h \to 0$), the secant line approaches the **tangent line** at P.

Example 1.3 (The famous P, Q picture with $f(x) = x^2$)**.** Let $f(x) = x^2$, $P = (1, 1)$, and $Q = (1 + h, (1 + h)^2)$. Then

$$m_{PQ} = 2 + h \longrightarrow 2 \quad (h \to 0).$$

So the tangent slope at $x = 1$ is 2.



1.3 Instantaneous rate of change

The **instantaneous rate of change** at $x = a$ is the limit of average rates of change:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

This number (when it exists) is the slope of the tangent line at $(a, f(a))$.

Example 1.4 (Average rate of change on a curvy function)**.** Let $f(x) = \ln x$ and compute the average rate of change on $[1, 1+h]$:

$$\frac{\ln(1+h) - \ln(1)}{h} = \frac{\ln(1+h)}{h}.$$

As $h \to 0$, this approaches the instantaneous rate of change of $\ln x$ at $x = 1$ (which will later be 1).

Remark. This chapter develops limits and continuity so that instantaneous rates of change (derivatives) can be defined and computed reliably.

2 Limits of a Function and Limit Laws (Practical First)

2.1 What a limit measures

A limit describes what $f(x)$ *approaches* as x approaches a value c. It does *not* require that $f(c)$ be defined, and even if it is defined, it does not require $f(c)$ to equal the limit.

2.2 A motivating table: behavior near a point

Consider

$$f(x) = \frac{x^3 - 1}{x - 1}.$$

At $x = 1$ the expression is not defined, but we can study values near 1.

x (left of 1)	$f(x)$	x (right of 1)	$f(x)$
0.9	2.71	1.1	3.31
0.99	2.9701	1.01	3.0301
0.999	2.997001	1.001	3.003001

This strongly suggests $f(x) \to 3$ as $x \to 1$, so we write

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = 3.$$

2.3 Algebraic simplification (removing a hole)

Factor:

$$x^3 - 1 = (x - 1)(x^2 + x + 1) \quad \Rightarrow \quad \frac{x^3 - 1}{x - 1} = x^2 + x + 1 \quad (x \neq 1).$$

Thus the limit becomes:

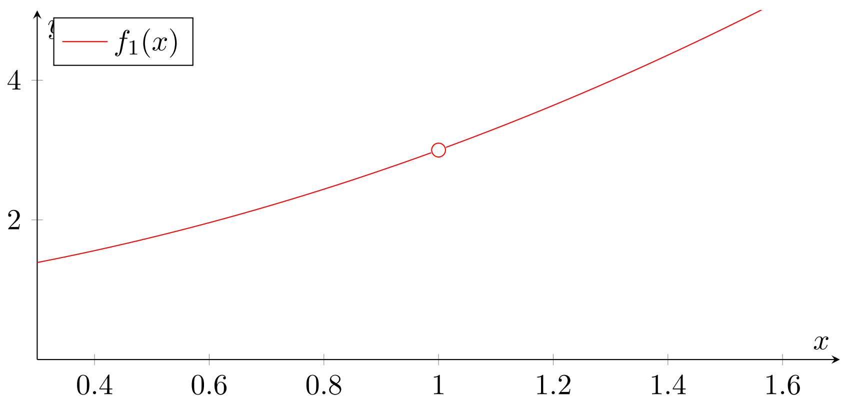
$$\lim_{x\to 1} (x^2 + x + 1) = 3.$$

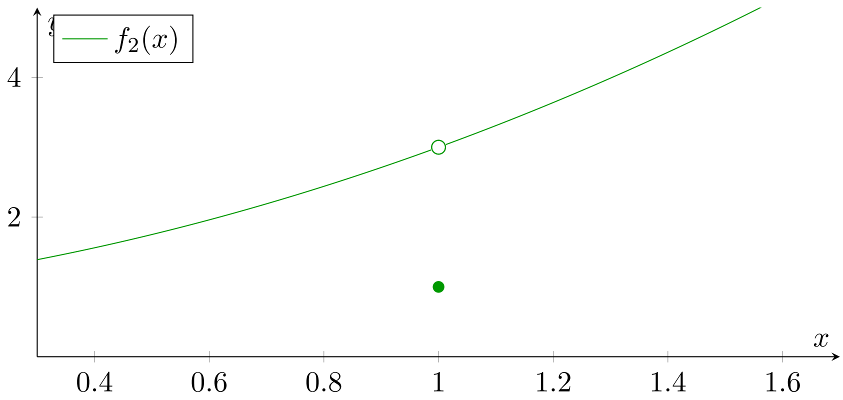
2.4 Three graphs: same limit, different point values

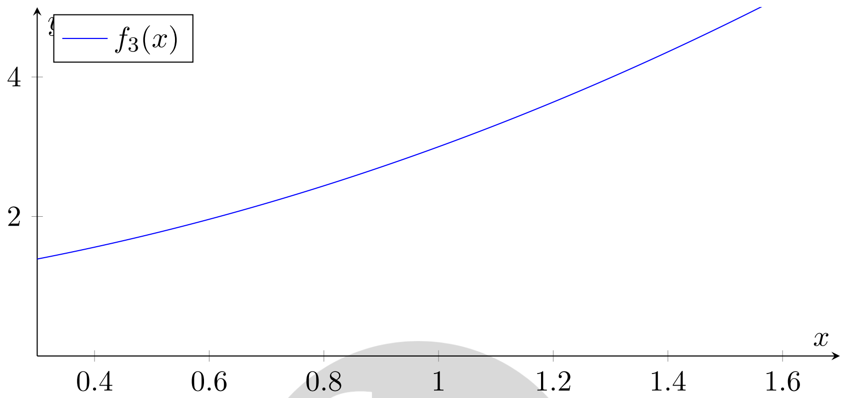
Define:

$$f_1(x) = \frac{x^3 - 1}{x - 1} \ (x \neq 1), \quad f_2(x) = \begin{cases} \frac{x^3-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}, \quad f_3(x) = x^2 + x + 1.$$

All satisfy $\lim_{x\to 1} f_i(x) = 3$, but their values at $x = 1$ differ.







2.5 Limit laws (Theorem)

Theorem 2.1 (Limit Laws)**.** Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ and k is a constant. Then:

1. (Sum) $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. (Difference) $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. (Constant multiple) $\lim_{x\to c}(kf(x)) = kL$
4. (Product) $\lim_{x\to c}(f(x)g(x)) = LM$
5. (Quotient) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$
6. (Power) $\lim_{x\to c}(f(x))^n = L^n$ for $n \in \mathbb{N}$
7. (Root) $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ (when defined in real numbers)

2.6 Polynomials and rational functions

Theorem 2.2 (Limits of polynomials)**.** If $p(x)$ is a polynomial, then for any $c \in \mathbb{R}$,

$$\lim_{x\to c} p(x) = p(c).$$

Theorem 2.3 (Limits of rational functions)**.** If $r(x) = \frac{p(x)}{q(x)}$ and $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = \frac{p(c)}{q(c)}.$$

Example 2.1 (Direct substitution)**.**

$$\lim_{x \to -2} \frac{x^3+1}{x^2+5} = \frac{-8+1}{4+5} = -\frac{7}{9}.$$

Example 2.2 (A rational function with a removable discontinuity)**.**

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3}(x+3) = 6.$$

The original function has a *hole* at $x = 3$, but the limit exists and is 6.

2.7 The Squeeze (Sandwich) Theorem

Theorem 2.4 (Squeeze Theorem)**.** If $f(x) \le g(x) \le h(x)$ for all x in some open interval containing c (except possibly at $x = c$), and if

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L,$$

then

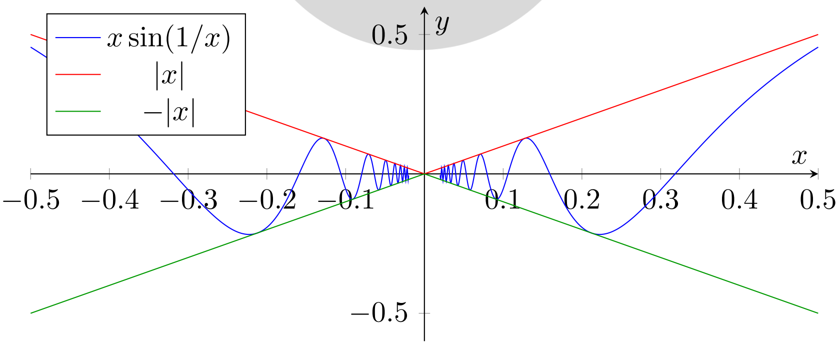
$$\lim_{x \to c} g(x) = L.$$

Example 2.3 (Oscillations squeezed to 0)**.** For all $x \neq 0$,

$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x|.$$

As $x \to 0$, both bounds go to 0, so

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$



2.8 Order and limits

Theorem 2.5 (Order is preserved under limits)**.** If $f(x) \le g(x)$ for all x in some open interval containing c (except possibly at $x = c$), and if the limits exist, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

3 The Precise Definition of a Limit (Epsilon–Delta)

3.1 The ε–δ definition

Definition 3.1 (ε–δ definition)**.** We write

$$\lim_{x \to c} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever

$$0 < |x - c| < \delta,$$

it follows that

$$|f(x) - L| < \varepsilon.$$

3.2 What the symbols mean (geometry)

- ε controls the vertical accuracy: we want $f(x)$ within $L \pm \varepsilon$.
- δ controls how close x must be to c: we restrict x within $c \pm \delta$.

The definition says: *no matter how small* ε is chosen, we can find a δ that makes the implication true.

3.3 A complete ε–δ proof (linear example)

Example 3.1 (Prove $\lim_{x \to 1}(2x + 1) = 3$)**.** Let $\varepsilon > 0$. We need $|2x + 1 - 3| < \varepsilon$ whenever $0 < |x - 1| < \delta$.

$$|2x + 1 - 3| = |2x - 2| = 2|x - 1|.$$

So it suffices to ensure $2|x - 1| < \varepsilon$, i.e. $|x - 1| < \varepsilon/2$. Choose $\delta = \varepsilon/2$. Then $0 < |x - 1| < \delta$ implies $|2x + 1 - 3| < \varepsilon$.

3.4 A nonlinear ε–δ proof (quadratic example)

Example 3.2 (Prove $\lim_{x \to 2} x^2 = 4$)**.** We want: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |x^2 - 4| < \varepsilon.$$

Factor:

$$|x^2 - 4| = |x - 2||x + 2|.$$

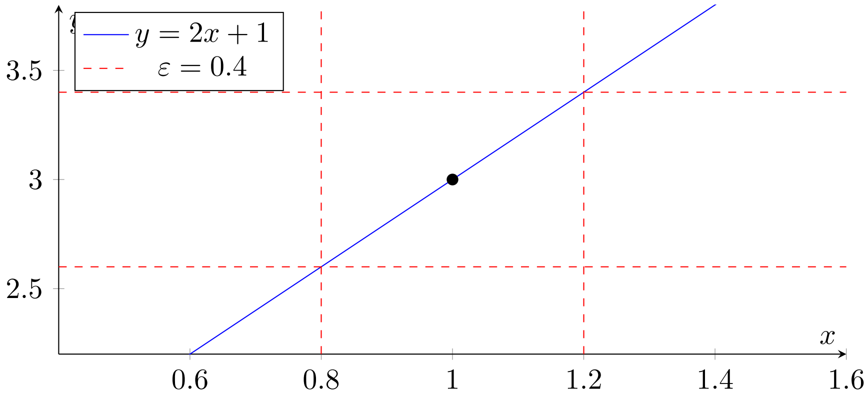
The term $|x + 2|$ depends on x, so we bound it by restricting x near 2. If we force $|x - 2| < 1$, then $1 < x < 3$, so $3 < x + 2 < 5$ and hence $|x + 2| < 5$. Thus, if $|x - 2| < 1$, then

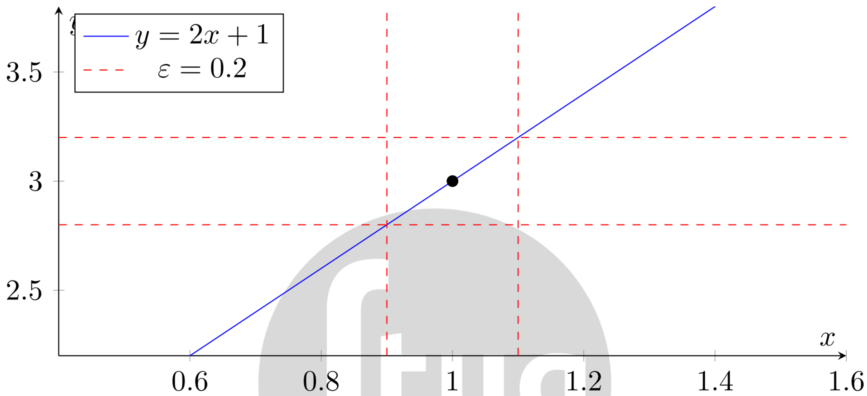
$$|x^2 - 4| = |x - 2||x + 2| < 5|x - 2|.$$

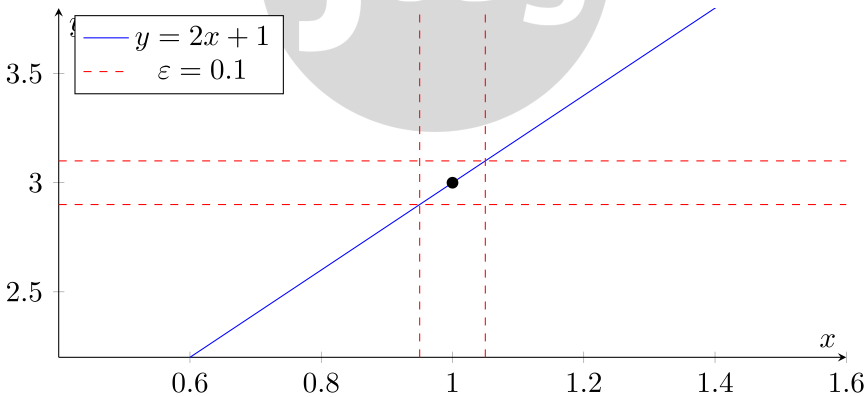
Now choose $\delta = \min\{1, \varepsilon/5\}$. Then $|x - 2| < \delta$ implies $|x^2 - 4| < \varepsilon$.

3.5 Epsilon bands shrinking (visual)

Below, we visualize $y = 2x + 1$ approaching $L = 3$ at $c = 1$ with smaller and smaller ε.







4 One-Sided Limits and Key Trigonometric Limits

4.1 Left-hand and right-hand limits

Definition 4.1 (One-sided limits)**.**

$$\lim_{x\to c^-} f(x) = L \quad \text{(approach from the left)}, \qquad \lim_{x\to c^+} f(x) = L \quad \text{(approach from the right)}.$$

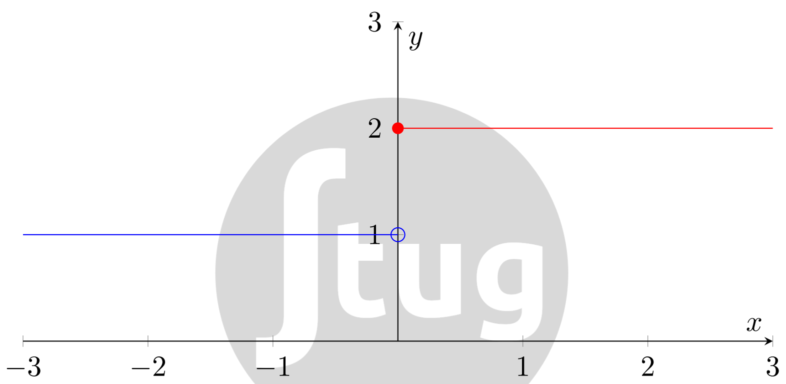
A two-sided limit exists if and only if both one-sided limits exist and are equal.

4.2 Piecewise examples (where one-sided limits differ)

Example 4.1 (Jump discontinuity)**.**

$$f(x) = \begin{cases} 1, & x < 0, \\ 2, & x \geq 0. \end{cases} \quad \Rightarrow \quad \lim_{x\to 0^-} f(x) = 1, \; \lim_{x\to 0^+} f(x) = 2,$$

so $\lim\limits_{x\to 0} f(x)$ does not exist.



4.3 One-sided limits where the function blows up

Example 4.2. For $f(x) = \dfrac{1}{x}$,

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

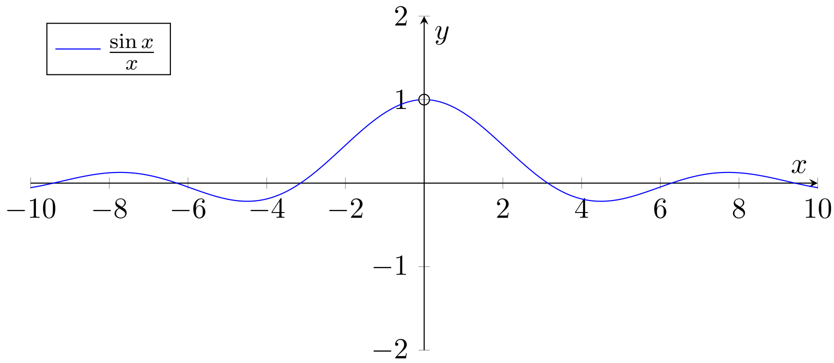
So the two-sided limit $\lim_{x\to 0} \frac{1}{x}$ does not exist.

4.4 The fundamental trig limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$

A central fact (proved geometrically or via squeeze) is:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

x (left)	$\frac{\sin x}{x}$	x (right)	$\frac{\sin x}{x}$
-0.5	0.958851	0.5	0.958851
-0.1	0.998334	0.1	0.998334
-0.01	0.999983	0.01	0.999983



Example 4.3.

$$\lim_{x\to 0} \frac{\sin(5x)}{x} = \lim_{x\to 0} 5 \cdot \frac{\sin(5x)}{5x} = 5.$$

Example 4.4.

$$\lim_{x\to 0} \frac{1-\cos x}{x}.$$

Multiply by the conjugate:

$$\frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right).$$

As $x \to 0$, $\frac{\sin x}{x} \to 1$ and $\sin x \to 0$, $\cos x \to 1$, so

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

5 Continuity

5.1 Continuity at a point

Definition 5.1 (Continuity at c)**.** A function f is **continuous at** c if:

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists,
3. $\lim_{x \to c} f(x) = f(c)$.

Equivalently: $\lim_{x \to c} f(x) = f(c)$.

5.2 A discontinuity gallery

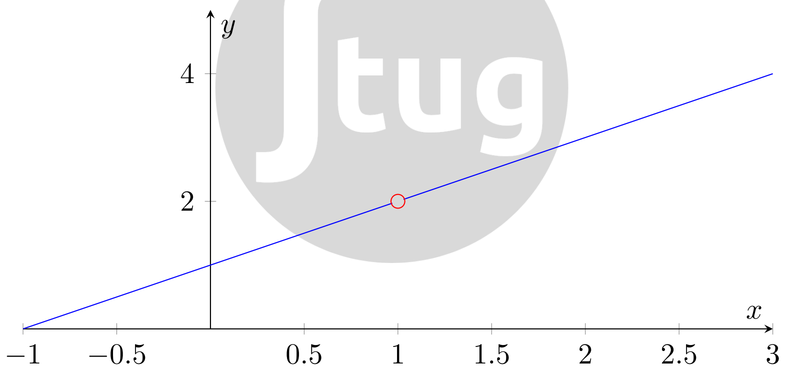
(A) Removable discontinuity

A typical removable discontinuity occurs when a factor cancels but the original function is undefined at the cancellation point.

Example 5.1.

$$f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1.$$

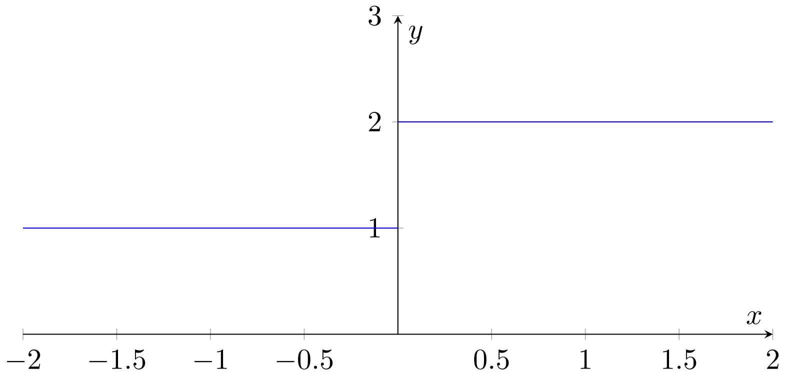
Since $x^2 - 1 = (x-1)(x+1)$, we have $f(x) = x + 1$ for $x \neq 1$. The limit as $x \to 1$ is 2, but $f(1)$ is not defined (a hole).



(B) Jump discontinuity

$$f(x) = \begin{cases} 1, & x < 0, \\ 2, & x \geq 0. \end{cases}$$

Left and right limits differ, so no two-sided limit.

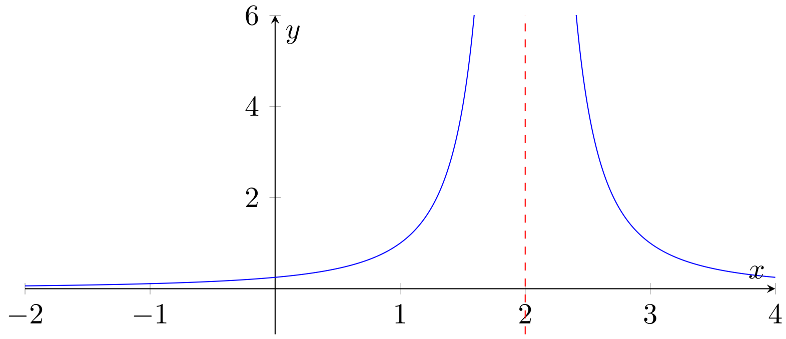


(C) Infinite discontinuity

When $f(x)$ goes to $\pm\infty$ near a point (vertical asymptote), the function is not continuous there.

Example 5.2.

$$f(x) = \frac{1}{(x-2)^2} \quad \Rightarrow \quad \lim_{x \to 2^{\pm}} f(x) = +\infty.$$



5.3 Right-continuity and left-continuity

Definition 5.2 (One-sided continuity)**.** f is **continuous from the right** at c if

$$\lim_{x \to c^+} f(x) = f(c),$$

and **continuous from the left** at c if

$$\lim_{x \to c^-} f(x) = f(c).$$

5.4 Continuity test

To check continuity at c:

1. Evaluate $f(c)$ (is it defined?).
2. Compute $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$.
3. If they are equal to each other and to $f(c)$, then f is continuous at c.

5.5 Algebra of continuous functions

Theorem 5.1 (Properties of continuous functions)**.** If f and g are continuous at c, then the following are continuous at c:

1. $f + g$
2. $f - g$
3. kf for $k \in \mathbb{R}$
4. fg
5. $\frac{f}{g}$ $(g(c) \neq 0)$
6. $(f)^n$ for $n \in \mathbb{N}$
7. $\sqrt[n]{f}$ (where defined)

5.6 Composition

Theorem 5.2 (Composition of continuous functions)**.** If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Example 5.3. Let $g(x) = x^2 + 1$ and $f(u) = \sqrt{u}$. Then g is continuous everywhere and f is continuous for $u \geq 0$. Since $g(x) \geq 1$, the composition $\sqrt{x^2 + 1}$ is continuous for all x.

5.7 Limits of continuous functions

Theorem 5.3 (Limits via continuity)**.** If f is continuous at c, then

$$\lim_{x \to c} f(x) = f(c).$$

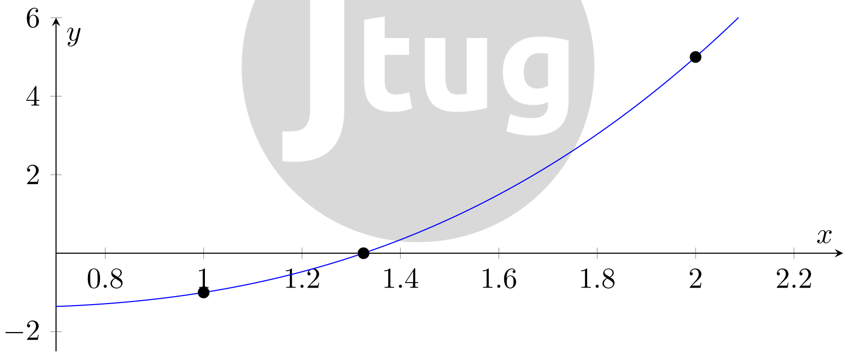
5.8 Intermediate Value Theorem (IVT)

Theorem 5.4 (Intermediate Value Theorem)**.** If f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = N$.

Example 5.4 (Existence of a root)**.** Show $x^3 - x - 1 = 0$ has a solution in $(1, 2)$.
Let $p(x) = x^3 - x - 1$, continuous everywhere.

$$p(1) = -1, \qquad p(2) = 5.$$

Since 0 is between -1 and 5, IVT guarantees some $c \in (1, 2)$ with $p(c) = 0$.



Example 5.5 (IVT used for solving $\cos x = x$ (existence))**.** Consider $f(x) = \cos x - x$, continuous everywhere.

$$f(0) = 1, \qquad f(1) = \cos 1 - 1 < 0.$$

So there exists $c \in (0, 1)$ with $\cos c = c$.

6 Limits Involving Infinity and Asymptotes

6.1 Limits as $x \to \infty$ and $x \to -\infty$

Definition 6.1 (Limits at infinity)**.** We say $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists M such that

$$x > M \Rightarrow |f(x) - L| < \varepsilon.$$

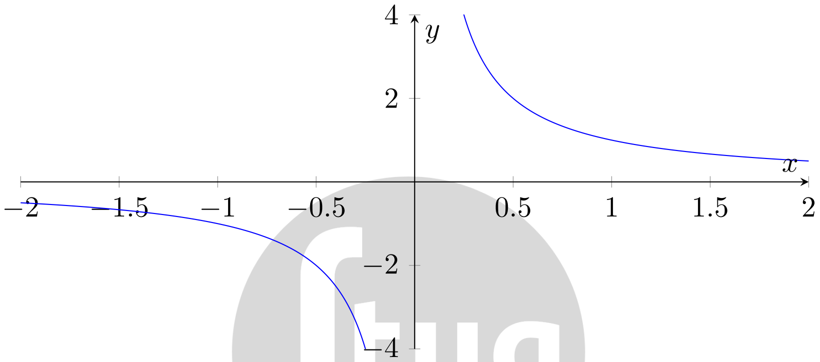
Similarly, $\lim_{x\to-\infty} f(x) = L$ if there exists M such that

$$x < M \Rightarrow |f(x) - L| < \varepsilon.$$

Example 6.1.

$$\lim_{x\to\infty} \frac{1}{x} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

From the graph we observe that as $x \to \pm\infty$, the function values approach 0.



6.2 Horizontal asymptotes

Definition 6.2 (Horizontal asymptote)**.** If $\lim_{x\to\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f as $x \to \infty$. If $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote as $x \to -\infty$.

6.3 Rational functions at infinity: degree rules

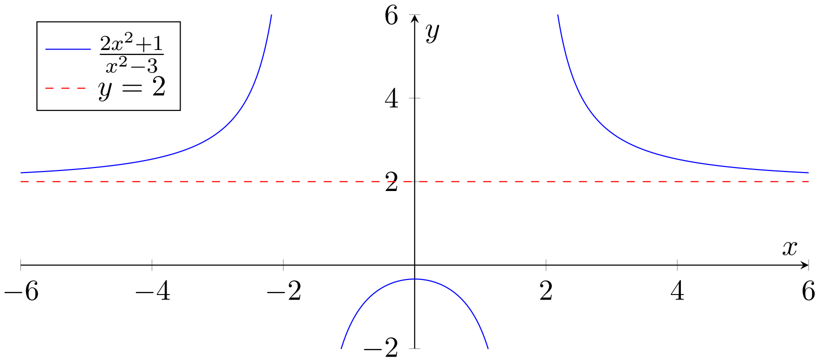
Let $r(x) = \dfrac{p(x)}{q(x)}$ where p, q are polynomials.

- If $\deg p < \deg q$, then $\lim_{x\to\pm\infty} r(x) = 0$.
- If $\deg p = \deg q$, then $\lim_{x\to\pm\infty} r(x)$ equals the ratio of leading coefficients.
- If $\deg p = \deg q + 1$, then $r(x)$ has a **slant (oblique) asymptote**.

Example 6.2 (Equal degrees)**.**

$$\lim_{x\to\infty} \frac{2x^2 + 1}{x^2 - 3} = \lim_{x\to\infty} \frac{2 + \frac{1}{x^2}}{1 - \frac{3}{x^2}} = 2.$$

So $y = 2$ is a horizontal asymptote.



6.4 Slant (oblique) asymptotes

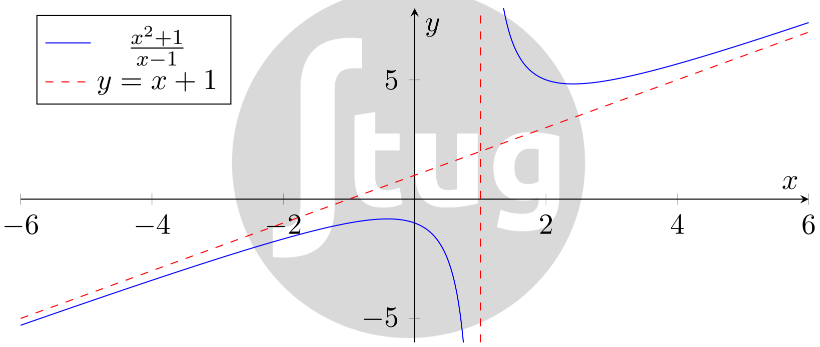
Example 6.3.

$$f(x) = \frac{x^2 + 1}{x - 1}.$$

Division gives

$$\frac{x^2 + 1}{x - 1} = x + 1 + \frac{2}{x - 1},$$

so the slant asymptote is $y = x + 1$.



6.5 Infinite limits and vertical asymptotes

Definition 6.3 (Infinite limit)**.** We write $\lim_{x \to c} f(x) = +\infty$ if for every $M > 0$ there exists $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow f(x) > M.$$

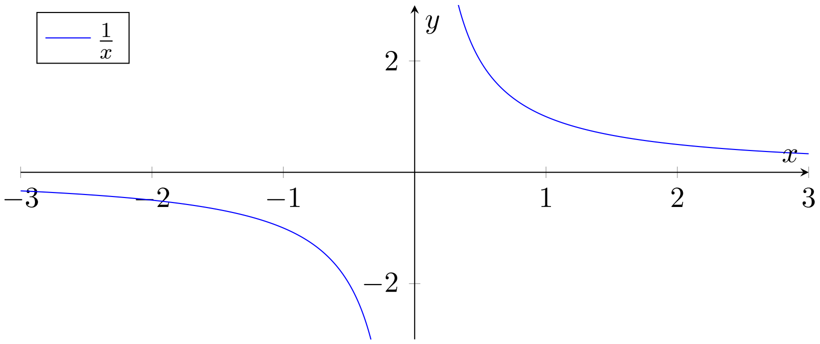
Similarly, $\lim_{x \to c} f(x) = -\infty$ if $f(x) < -M$ near c.

Definition 6.4 (Vertical asymptote)**.** If $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$, then $x = c$ is a vertical asymptote.

Example 6.4 (The sign matters: $\frac{1}{x}$ at 0)**.**

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

So there is no two-sided infinite limit at 0.



Example 6.5 (Both sides to $+\infty$)**.**

$$f(x) = \frac{1}{(x-2)^2} \quad \Rightarrow \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = +\infty,$$

so $x = 2$ is a vertical asymptote.