# Chapter 2: Derivatives I

## Contents

# 1 Tangents and the Derivative at a Point

One of the central problems of calculus is determining the slope of a curve at a particular point.

Consider a function $y = f(x)$ and a point on the graph

$$P(x_0, f(x_0)).$$

We want to find the slope of the line that just "touches" the curve at this point. This line is called the **tangent line**.

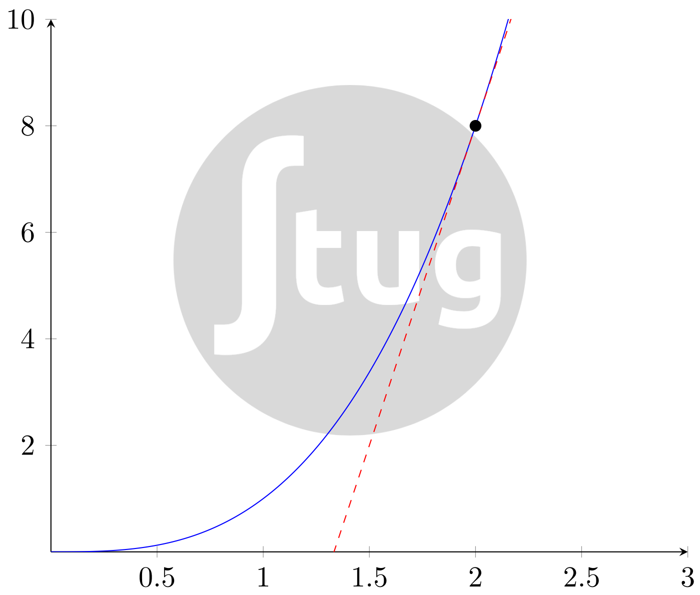
To approximate this slope we consider another point on the graph

$$Q(x_0 + h, f(x_0 + h)).$$

The slope of the line through $P$ and $Q$ is

$$\frac{f(x_0 + h) - f(x_0)}{h}.$$

This line is called a **secant line**. If we move the point $Q$ closer to $P$, the secant line begins to resemble the tangent line.




**Definition 1.1.** The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$ is

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that this limit exists.

**Example 1.1.** Find the slope of the curve $f(x) = x^3$ at the point $x = 2$.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{(2+h)^3 - 8}{h} \\
&= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
&= \lim_{h\to 0} \frac{h(12 + 6h + h^2)}{h} \\
&= \lim_{h\to 0} (12 + 6h + h^2) \\
&= 12
\end{aligned}$$

Thus the tangent slope is 12.

## 1.1 Rates of Change

The slope of the tangent line represents how fast the function changes.

**Definition 1.2.** The derivative of a function $f$ at $x_0$ is

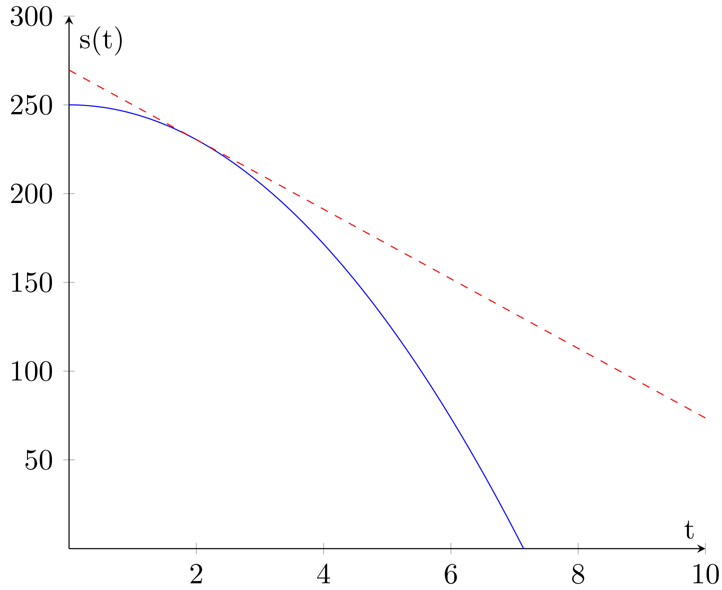
$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This quantity represents the **instantaneous rate of change** of the function.

**Example 1.2.** Suppose that current height of an object is 250 meter. If the objects height over time (seconds) is $s(t) = 250 - 4.9t^2$ then find the velocity of the object at $t = 2$.

Then velocity must be:

$$\begin{aligned}
s'(2) &= \lim_{h\to 0} \frac{s(2+h) - s(2)}{h} \\
&= \lim_{h\to 0} \frac{250 - 4.9(2+h)^2 - 250 + 4.9(2)^2}{h} \\
&= \lim_{h\to 0} \frac{-4.9(4 + 4h + h^2) + 4.9(4)}{h} \\
&= \lim_{h\to 0} \frac{-4.9h(4+h)}{h} \\
&= \lim_{h\to 0} -4.9(4+h) \\
&= -19.6
\end{aligned}$$




The slope of the curve at a point represents the object's velocity at that instant.

## 2 The Derivative as a Function

Previously we defined the derivative at a single point. But the derivative can also be defined as an entire function.

**Definition 2.1.** The derivative of $f$ is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

An equivalent definition is

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

## 2.1 Calculating Derivatives from the Definition

Calculating the derivative of a function is called "differentiation" and oftenly use the notations: $\frac{d}{dx}f(x)$, $f'(x)$.

**Example 2.1.** Differentiate the function $f(x) = x^2 + 3x$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 3h + 2xh}{h} \\
&= \lim_{h \to 0} \frac{h(h + 3 + 2x)}{h} \\
&= \lim_{h \to 0} (h + 3 + 2x) \\
&= 2x + 3
\end{aligned}$$



## 2.2 Derivative Notations

Common notations include

| Lagrange | Leibniz | Newton | D Notation |
|---|---|---|---|
| $f'(x)$, $y'$ | $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$, $\frac{df(x)}{dx}$ | $\dot{x}$, $\dot{y}$, $\dot{z}$ | $(Df)(x), Df(x), D_x f$ |

## 2.3 One-Sided Derivatives

Like limits, derivatives can be defined from left or right side of the differentiated point.

$$\begin{aligned}
f'(a^-) &= \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \text{ Left-hand derivative at } a \\
f'(a^+) &= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \text{ Right-hand derivative at } a
\end{aligned}$$

We know that if the one sided limits exists on a point and are equal we can say that limit exists on that point, similarly here if the one sided derivatives exists and equal we can say the point is differentiable.

**Example 2.2.** Show that for the function $f(x) = |x-2|$, derivative does not exist at the point $x = 2$.

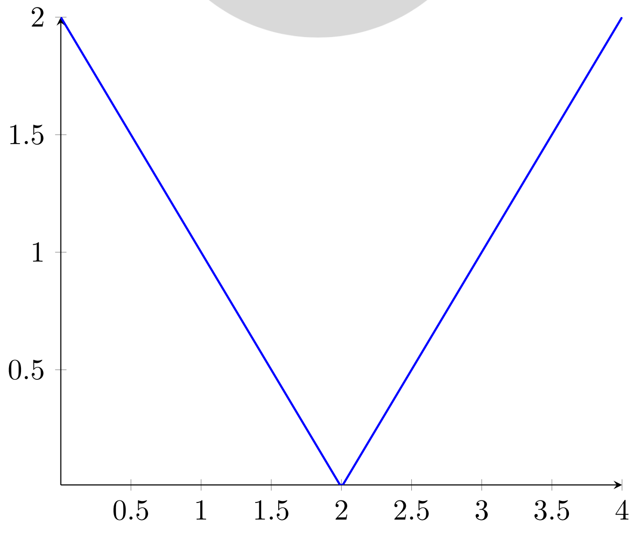
Left-hand derivative:

$$\begin{aligned} f'(2^-) &= \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0^-} \frac{|2+h-2| - |2-2|}{h} \\ &= \lim_{h \to 0^-} \frac{|h|}{h} \\ &= \lim_{h \to 0^-} \frac{-h}{h} && (|h| = -h \text{ when } h < 0) \\ &= -1 \end{aligned}$$

Right-hand derivative:

$$\begin{aligned} f'(2^+) &= \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0^+} \frac{|2+h-2| - |2-2|}{h} \\ &= \lim_{h \to 0^+} \frac{|h|}{h} \\ &= \lim_{h \to 0^+} \frac{h}{h} && (|h| = h \text{ when } h > 0) \\ &= 1 \end{aligned}$$

$f'(2^-) \neq f'(2^+)$ Therefore the derivative does not exist.




Here we can see the sharp change on the point $x = 2$ on the graph of $f(x) = |x-2|$.

### 2.4 Differentiability and Continuity

**Theorem 2.1** (Differentiability Implies Continuity)**.** If $f$ has a derivative at $x = c$, then $f$ is continuous at $x = c$.

Note that the rule does not apply for reverse, a function that is continuous at a point does not mean that the function can be differentiable at that point.

# 3 Differentiation Rules

The definition of the derivative allows us to compute derivatives directly. However this process can be lengthy. Fortunately several useful rules simplify calculations.

### 3.1 Derivative of a Constant

If a function $f$ has the constant value $f(x) = c$ then we would say:

$$\frac{d}{dx}f(x) = \frac{d}{dx}c = 0$$

We can prove it easily by:

$$\begin{aligned}\frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= 0\end{aligned}$$

### 3.2 Derivative of a Positive Integer Power

The positive integer rule is one of the most used formulas to differentiate. The rule suggests that for a function $f(x) = x^n$ where $n$ is an positive integer:

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^n = nx^{n-1}$$

For proof we use factorization identity:

$$z^n - x^n = (z-x)(z^{n-1} + z^{n-2}x + \cdots + x^{n-1}$$

For the positive integer rule, we will use the alternative definition to prove it:

$$\begin{aligned}\frac{d}{dx}f(x) &= \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\\ &= \lim_{z\to x}\frac{z^n-x^n}{z-x}\\ &= \lim_{z\to x}\frac{(z-x)(z^{n-1}+z^{n-2}x+\cdots+zx^{n-2}+x^{n-1})}{z-x} && \text{(Use factorization identity)}\\ &= \lim_{z\to x}(z^{n-1}+z^{n-2}x+\cdots+zx^{n-2}+x^{n-1})\\ &= nx^{n-1}\end{aligned}$$

Power rule can be generalized for any $n \in \mathbb{R}$ where $x^n$ and $x^{(}n-1)$ are defined.

## 3.3 Constant Multiple Rule

Constant multiple rule says that for any function $f(x) = cu(x)$, where $u(x)$ is also a function, we can do that:

$$\frac{d}{dx}f(x) = \frac{d}{dx}cu(x) = c\frac{d}{dx}u(x)$$

Proof:

$$\begin{aligned}
\frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} \\
&= \lim_{h\to 0}\frac{c[u(x+h)-u(x)]}{h} \\
&= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \\
&= c\frac{d}{dx}u(x)
\end{aligned}$$

## 3.4 Sum Rule

If two functions $f(x)$ and $g(x)$ differentiable where $f(x)+g(x)$ is also differentiable then we can apply a rule like:

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof:

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \frac{d}{dx}f(x) + \frac{d}{dx}g(x)
\end{aligned}$$

### 3.5 Product Rule

Derivative of sums are the sum of derivatives of the functions but unfortunately we can't say same for the derivative of products, for functions $u$ and $v$ which both are functions of $x$, we can apply a rule like this:

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

Proof:

$$\begin{aligned}
\frac{d}{dx}uv &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} [u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}] \\
&= \lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u\frac{dv}{dx} + v\frac{du}{dx}
\end{aligned}$$

### 3.6 Quotient Rule

We can easily calculate the derivative of forms $u(x)/v(x)$ with the quotient rule:

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Proof:

$$\begin{aligned}
\frac{d}{dx}\frac{u}{v} &= \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\
&= \lim_{h\to 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} \\
&= \frac{v(x) \lim\limits_{h\to 0} \frac{u(x+h) - u(x)}{h} - u(x) \lim\limits_{h\to 0} \frac{v(x+h) - v(x)}{h}}{\lim\limits_{h\to 0} v(x+h)v(x)} \\
&= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}
\end{aligned}$$

**Example 3.1.** Find the derivative of function $f(x) = \frac{x^3+2}{x}$.

$$
\begin{aligned}
f'(x) &= \frac{x(x^3+2)' - (x^3+2)(x)'}{(x)^2} \\
&= \frac{x(3x^2+0) - (x^3+2)1}{x^2} \\
&= \frac{3x^3 - x^3 - 2}{x^2} \\
&= \frac{2x^3 - 2}{x^2}
\end{aligned}
$$

# 4 The Derivative as a Rate of Change

The derivative measures how one quantity changes relative to another.

**Definition 4.1.** The instantaneous rate of change of $f$ with respect to $x$ at $x_0$ is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

## 4.1 Motion Along a Line

Let's suppose an object is moving along a coordinate line (a $s$-axis), we'd call the position on the line of the object over time $s(t)$.

Then the displacement of the object over time interval from $t$ to $\Delta t$ would be:

$$\Delta s = s(t + \Delta t) - s(t)$$

Average velocity of an object is the object's displacement over time and we can calculate it simply with:

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

However velocity (or instantaneous velocity) of an object is the derivative of position with respect to time which is oftenly denoted by $v$:

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

Speed is the absolute value of velocity:

$$|v(t)|$$

So we can say that the velocity of an object is directional (a vector) meanwhile speed of an object is magnitude of the velocity.

**Example 4.1.** Assume that a car's position over time is $p_c(t) = -15t^2 + 150$. Find the speed of the car at time $t = 3$.

Velocity of the car is:

$$p_c'(t) = -30t$$

At $t = 3$:

$$-30(3) = -90$$

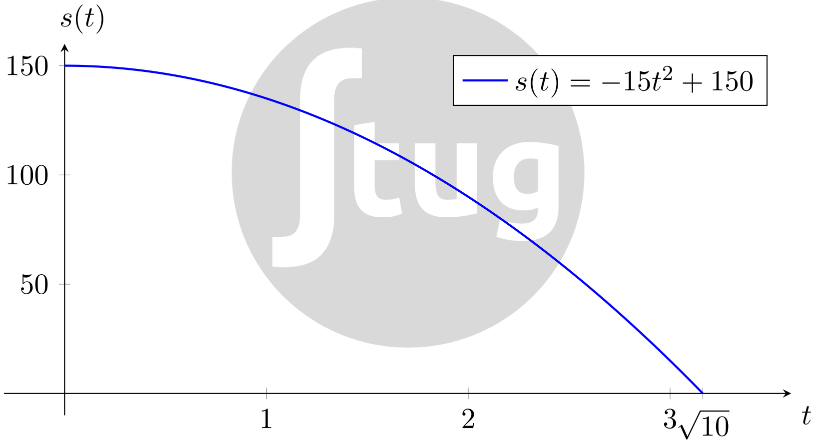
Speed is the absolute value of velocity so our answer is:

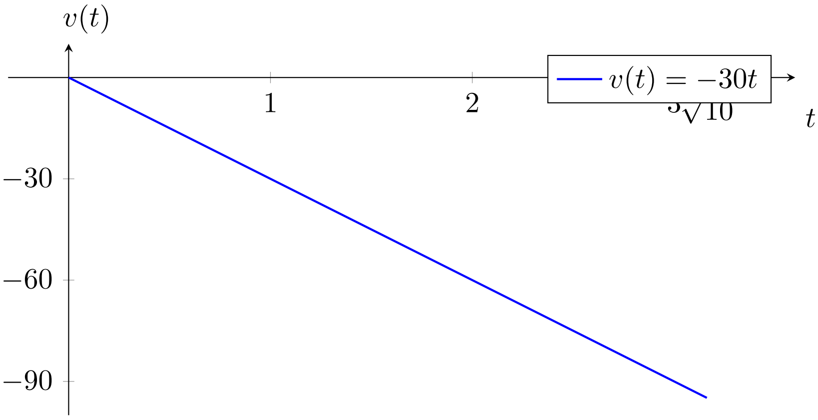
$$|-90| = 90$$

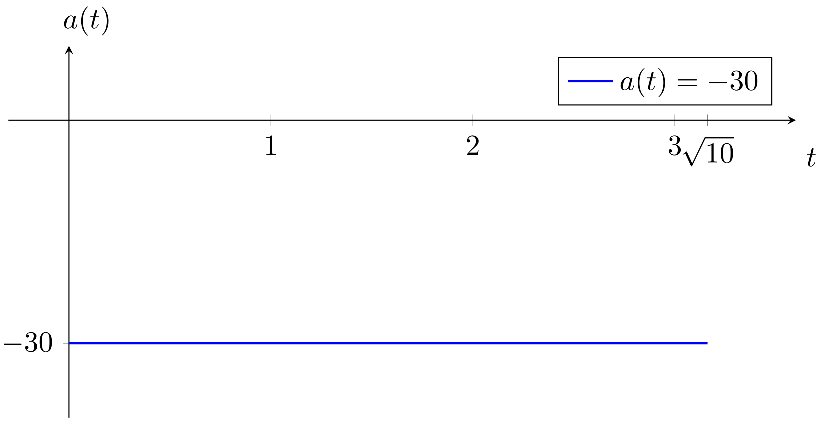
Acceleration is the derivative of velocity with respect to time and tells us how the velocity changes over time:

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

**Example 4.2.** Show the graphs of position $s(t) = -15t^2 + 150$, velocity $v(t) = s'(t)$ and acceleration $a(t) = v'(t)$ functions.






## 5 Derivatives of Trigonometric Functions

We now compute derivatives of trigonometric functions using the definition.

### 5.1 Derivative of Sine Function

We will use the angle sum identity of sine function:

$$\sin(x+h) = \sin x \cos h + \sin h \cos x$$

With that identity, we can differentiate sine function like in the following:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\sin h \cos x}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x(0) + \cos x(1) \\
&= \cos x
\end{aligned}$$

So we ended up in an equation like that:

$$\frac{d}{dx}\sin x = \cos x$$

### 5.2 Derivative of Cosine Function

Similarly we can use the angle sum identity of cosine function:

$$\cos(x + h) = \cos x \cos h - \sin x \sin h$$

So differentiating cosine function is:

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h} \\
&= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x(0) - \sin x(1) \\
&= -\sin x
\end{aligned}$$

And we found the derivative of cosine function:

$$\frac{d}{dx}\cos x = -\sin x$$

### 5.3 Simple Harmonic Motion

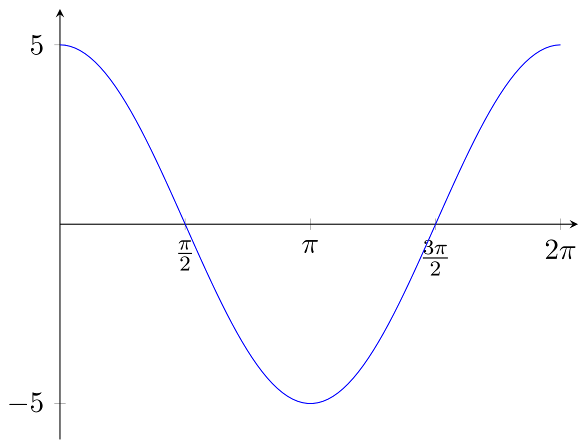
Trigonometric functions naturally arise when describing periodic motion. One of the most important examples is simple harmonic motion, which models situations where an object moves back and forth around an equilibrium position.

Such motion can be described by a function of the form

$$s(t) = A\cos(\omega t),$$

where $A$ represents the amplitude of the motion and $\omega$ determines how rapidly the motion oscillates.

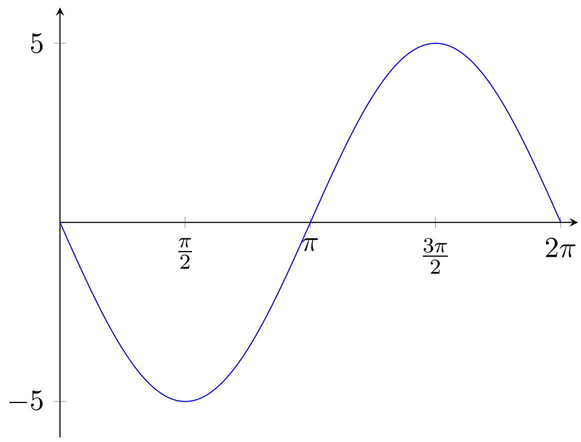
For example position of an object with simple harmonic motion over time for $A = 5$ and $\omega = 1$ changes like in the following graph:



We can find its velocity simply by differentiating the position function.

$$s'(t) = v(t) = -A\sin(\omega t)$$

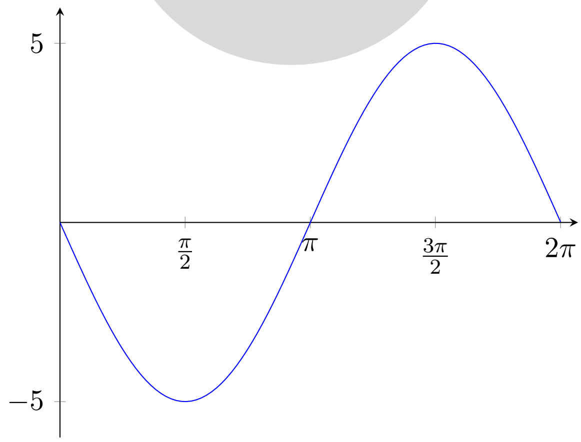
And for our example the graph of the velocity would be:




And taking the derivative of the velocity would give us the acceleration.

$$v'(t) = a(t) = -A\cos(\omega t)$$

The acceleration graph of the example:



### 5.4 Other Trigonometric Derivatives

Other trigonometric derivatives can be calculated with the differentiation rules as we already know sine and cosine derivatives.

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

### 5.5 Inverse Trigonometric Functions

It's important to know those derivatives for hard to compute integrals in the future.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

## Exercises

1. Find the slope of the tangent line to the function $y = (x+2)^2$ at $x = 2$.
2. Use definition of derivatives to find the derivative of function $f(x) = 2x + 1$.
3. Find the derivative of the function $f(x) = \frac{x^5+2x}{5}$.
4. What's the speed of the function $f(t) = -10t^2$ at $t = 1$.
5. Prove that $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$.