Chapter 3: Derivatives II

Contents

1 The Chain Rule

1.1 Derivative of a Composite Function

Many functions are built by placing one function inside another. For example,

$$y = \sin(x^2)$$

is a composition of

$$g(x) = x^2, \qquad f(u) = \sin u$$

Thus,

$$y = f(g(x))$$

The derivative of such a function depends on both the outer function and the inner function.

Example 1.1. Consider

$$y = (x^2 + 1)^5.$$

This is not simply a power of x. It is a power of the function $x^2 + 1$. We can write

$$u = x^2 + 1, \qquad y = u^5$$

Changing x changes u, and changing u changes y. The chain rule describes this connection.

1.2 Chain Rule Theorem

Theorem 1.1 (Chain Rule)**.** If $y = f(u)$ and $u = g(x)$ are differentiable functions, then the derivative of

$$y = f(g(x))$$

is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Equivalently,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Note that the expression $\frac{d}{dx}$ is not a fraction but an operator, however may be treated as a fraction in most cases.

Example 1.2. Differentiate $y = (x^2 + 1)^5$.

We can let $u = x^2 + 1$, then $y = u^5$, so:

$$\frac{dy}{du} = 5u^4, \qquad \frac{du}{dx} = 2x$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 5u^4(2x) \\ &= 10x(x^2 + 1)^4\end{aligned}$$

1.3 Repeated Use of the Chain Rule

Some functions contain several layers of composition. In that case, the chain rule must be applied repeatedly.

Example 1.3. Differentiate $y = \sin^3(x^2 + 1)$.

Let $u = \sin(x^2 + 1)$ and $v = x^2 + 1$. Then we have:

$$y = u^3$$

So

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} \\ &= 3u^2 \cos(v)2x \\ &= 3(\sin(v))^2 \cos(x^2 + 1)2x \\ &= 6x \sin^2(x^2 + 1) \cos(x^2 + 1)\end{aligned}$$

1.4 The Chain Rule with Power of a Function

A very common form is

$$y = [u(x)]^n.$$

By the chain rule,

$$\frac{dy}{dx} = n[u(x)]^{n-1}u'(x).$$

Example 1.4. Differentiate $y = (3x^2 - 4x + 1)^7$.

Let $u = 3x^2 - 4x + 1$, then we can differentiate y like in the following:

$$\begin{aligned}\frac{dy}{dx} &= (u^7)'u' \\ &= 7u^6(3x^2 - 4x + 1)' \\ &= 7(3x^2 - 4x + 1)^6(6x - 4)\end{aligned}$$

Example 1.5. Differentiate $y = \sqrt{x^2 + 5}$.

$$\begin{aligned}\frac{dy}{dx} &= (\sqrt{x^2 + 5})' \\ &= [(x^2 + 5)^{1/2}]' \\ &= \frac{1}{2}(x^2 + 5)^{-1/2}(2x) \\ &= \frac{x}{\sqrt{x^2 + 5}}\end{aligned}$$

2 Implicit Differentiation

2.1 Implicitly Defined Functions

So far, functions were mostly written in the form

$$y = f(x).$$

Such equations define y explicitly in terms of x. However, many curves are given by equations involving both x and y, such as

$$x^2 + y^2 = 1.$$

This equation does not define a single function $y = f(x)$ on the whole interval $[-1, 1]$, but parts of the curve can still behave like functions.

When y is determined by an equation involving both x and y, we say y is implicitly defined.

Definition 2.1 (Implicit Differentiation)**.** Implicit differentiation is the method of differentiating both sides of an equation involving x and y, treating y as a differentiable function of x.

Since y depends on x, we use the chain rule:

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}.$$

Example 2.1. Find $\frac{dy}{dx}$ for $x^2 + y^2 = 1$.

$$
\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}1 \\
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\
2x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{-2x}{2y} \\
\frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}
$$

Example 2.2. Find $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$.

$$
\begin{aligned}
x^3 + y^3 &= 6xy \\
(x^3 + y^3)' &= (6xy)' \\
3x^2 + 3y^2y' &= 6y + 6xy' && \text{(Divide all by 3)} \\
x^2 + y^2y' &= 2y + 2xy' \\
y^2y' - 2xy' &= 2y - x^2 \\
y'(y^2 - 2x) &= 2y - x^2 \\
y' &= \frac{2y - x^2}{y^2 - 2x}
\end{aligned}
$$

2.2 Derivatives of Higher Order

For implicitly defined functions, higher derivatives can also be found by differentiating again.

Definition 2.2 (Higher-Order Derivatives)**.** If y is a differentiable function of x, then

$$\frac{dy}{dx} = y'$$

is the first derivative. The derivative of y' is the second derivative:

$$\frac{d^2y}{dx^2} = y''.$$

Further derivatives are denoted by

$$y''', \qquad y^{(4)}, \qquad \frac{d^ny}{dx^n}.$$

Example 2.3. For $x^2 + y^2 = 1$, we found:

$$y' = -\frac{x}{y}.$$

Find y''.

$$\begin{aligned}
y' &= -\frac{x}{y} \\
y'' &= (-\frac{x}{y})' \\
y'' &= -(\frac{x}{y})' \\
y'' &= -(\frac{1}{y} - \frac{xy'}{y^2}) \\
y'' &= -\frac{1}{y} + \frac{xy'}{y^2} && \text{(Replace y')} \\
y'' &= -\frac{1}{y} + \frac{x(-\frac{x}{y})}{y^2} \\
y'' &= -\frac{1}{y} - \frac{x}{y^3} \\
y'' &= -\frac{y^2 + x}{y^3}
\end{aligned}$$

3 Related Rates

3.1 Related Rates Equations

In related rates problems, several quantities change with time, and an equation connects them.

The usual process is:

1. Identify the changing quantities.
2. Write an equation relating them.
3. Differentiate both sides with respect to time.
4. Substitute known values.
5. Solve for the unknown rate.

Example 3.1. The radius of a circle increases at a rate of $\frac{dr}{dt} = 2\text{cm/s}$. Find the rate of change of the area when $r = 5$ cm.

We know that area of a circle is $A = \pi r^2$. Both A and r are dependent on t here, so we can differentiate both sides respect to t.

$$A(t) = \pi r^2$$
$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2)$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Now we can replace $r = 5$ and $\frac{dr}{dt} = 2$:

$$\frac{dA}{dt} = 2\pi(5)(2)$$
$$\frac{dA}{dt} = 20\pi \text{ cm}^2/s$$

4 Linearization and Differentials

4.1 Linearization

A differentiable function can be approximated near a point by its tangent line. This approximation is useful when the function is difficult to compute directly.

If f is differentiable at $x = a$, then the tangent line at $x = a$ is

$$y = f(a) + f'(a)(x - a).$$

This tangent line is called the **linearization** of f at a.

Definition 4.1 (Standard Linear Approximation)**.** The standard linear approximation of f at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a).$$

For x close to a,

$$f(x) \approx L(x).$$

Example 4.1. Approximate $\sqrt{4.1}$.

Let $f(x) = \sqrt{x}$ and choose $a = 4$, because $\sqrt{4} = 2$ is easy to compute.

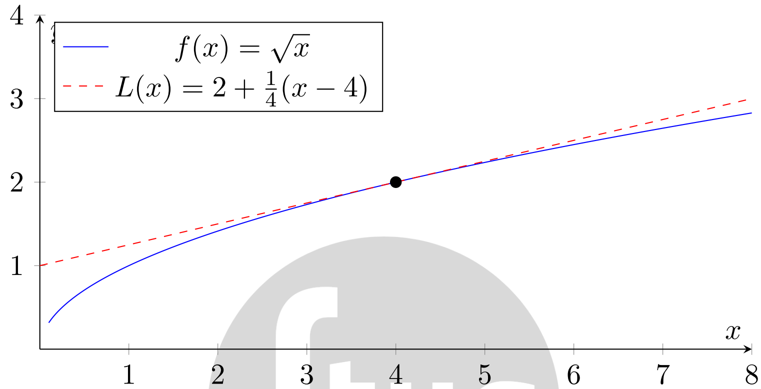
$$f'(x) = \frac{1}{2\sqrt{x}} \implies f'(4) = \frac{1}{4}$$

The linearization at $a = 4$ is

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

Therefore,

$$\sqrt{4.1} = f(4.1) \approx L(4.1) = 2 + \frac{1}{4}(0.1) = 2.025.$$



4.2 Differentials

Differentials provide another way to express small changes.

If

$$y = f(x),$$

then a small change in x is denoted by

$$dx.$$

The corresponding linear approximation to the change in y is denoted by

$$dy.$$

Definition 4.2 (Differential)**.** If $y = f(x)$ is differentiable, then the differential dy is defined by

$$dy = f'(x)\,dx.$$

Here dx is an independent small change in x, while dy is the corresponding linear approximation to the change in y.

The actual change in y is

$$\Delta y = f(x + \Delta x) - f(x).$$

For small changes,

$$\Delta y \approx dy.$$

4.3 Estimating with Differentials

Example 4.2. Use differentials to estimate the change in $y = x^3$ when x changes from 2 to 2.01.

Here

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ dy &= f'(x)dx \end{aligned}$$

At $x = 2$,

$$f'(2) = 12.$$

The change in x is

$$dx = 0.01$$

Thus,

$$\begin{aligned} dy &= f'(2)dx \\ &= 12(0.01) \\ &= 0.12 \end{aligned}$$

So

$$(2.01)^3 - 2^3 \approx 0.12.$$

4.4 Error in Differential Approximation

The differential dy approximates the actual change Δy, but they are not always equal.

$$\Delta y = f(x + \Delta x) - f(x), \qquad dy = f'(x)\Delta x.$$

The error is

$$\text{Error} = \Delta y - dy.$$

Example 4.3. For $f(x) = x^2$, estimate the error when $x = 3$ and $\Delta x = 0.1$.

Actual change:

$$\begin{aligned}\Delta y &= f(3.1) - f(3)\\ &= 3.1^2 - 3^2\\ &= 9.61 - 9\\ &= 0.61\end{aligned}$$

Differential approximation:

$$\begin{aligned}dy &= f'(3)\Delta x\\ &= 6(0.1)\\ &= 0.6\end{aligned}$$

Thus,

$$\text{Error} = 0.61 - 0.6 = 0.01.$$

4.5 Sensitivity to Change

Differentials also show sensitivity. If

$$dy = f'(x)dx,$$

then a larger value of $|f'(x)|$ means that small changes in x cause larger changes in y.

Example 4.4. Compare the sensitivity of $f(x) = x^2$ at $x = 1$ and $x = 10$.

Since

$$f'(x) = 2x,$$

we have

$$f'(1) = 2, \qquad f'(10) = 20.$$

A small change in x has a much larger effect near $x = 10$ than near $x = 1$.

5 Anti-derivatives

Derivatives answer the question:

$$\text{If we know } f(x), \text{ what is } f'(x)?$$

Anti-derivatives reverse this question:

$$\text{If we know } f'(x), \text{ what could } f(x) \text{ be?}$$

Definition 5.1 (Anti-derivative)**.** A function F is called an anti-derivative of f on an interval if

$$F'(x) = f(x)$$

for every x in that interval.

Example 5.1. Find an anti-derivative of $f(x) = 2x$.

We want a function F such that

$$F'(x) = 2x.$$

Since

$$\frac{d}{dx}(x^2) = 2x,$$

one anti-derivative is

$$F(x) = x^2.$$

In this example we only found an unique anti-derivative of $2x$ but since the derivative of a constant is 0, there are not a single anti-derivative but infinately many so we show it like in the following:

$$F(x) = x^2 + C.$$

C here is a constant.

Example 5.2. Find the general anti-derivative of

$$f(x) = 3x^2.$$

Since

$$\frac{d}{dx}(x^3) = 3x^2,$$

the general anti-derivative is

$$F(x) = x^3 + C.$$

Exercises

1. Differentiate $y = \cos(x^2 + 3x)$.
2. Find $\dfrac{dy}{dx}$ implicitly if $x^2 + xy + y^2 = 7$.
3. The radius of a sphere increases at $\dfrac{dr}{dt} = 3$ cm/s. Find $\dfrac{dV}{dt}$ when $r = 4$, where $V = \frac{4}{3}\pi r^3$.
4. Use linearization to approximate $\sqrt{9.2}$.
5. Find the anti-derivative of $f(x) = 4x^3 + 3$.